Period 3, March 3, 2025

$$\int_1^3 \frac{e^{3x}}{e^{3x}-1}dx = \int \frac{\cancel{e^{3x}}}{u} \cdot \frac{du}{3\cancel{e^{3x}}} = \frac{1}{3}\int_{e^3-1}^{e^9-1} \frac{du}{u}$$

$u = e^{3x}-1$

$du = 3e^{3x}dx + 0$

$\frac{du}{3e^{3x}} = dx$

$e^{3\cdot 1}-1 = e^3-1$

$e^{3\cdot 3}-1 = e^9-1$

$y = e^{3x} \Rightarrow y = e^u$

$u = 3x$

$\frac{du}{dx} = 3$

$\frac{dy}{du} = e^u$

$\frac{du}{dx}\cdot\frac{dy}{du} = \frac{dy}{dx}$

$3\cdot e^u = 3e^{3x} = \frac{dy}{dx}$

$$\frac{1}{3}\cdot \ln^u \Big[= \frac{1}{3}(\ln|e^9-1| - \ln|e^3-1|)\Big]$$

$a^3-b^3 =$

$(a-b)(a^2+ab+b^2)$

$$\frac{1}{3}\ln|e^{3x}-1|\Big[_1^3 = \frac{1}{3}\ln|e^{3\cdot 3}-1| - \frac{1}{3}\ln|e^{3\cdot 1}-1|$$

e^9-1

$(e^3)^3-1^3$

$(e^3-1)((e^3)^2+1\cdot e^3+1^2)$

$$\frac{1}{3}\Big[\ln|e^9-1| - \ln|e^3-1|\Big] = \frac{1}{3}\Big[\ln \frac{e^9-1}{e^3-1}\Big] = \frac{1}{3}\ln|e^6+e^3+1|$$

$$\frac{1}{3}\ln \frac{\cancel{(e^3-1)}(e^6+e^3+1)}{\cancel{(e^3-1)}}$$

$$\ln\sqrt[3]{\frac{e^9-1}{e^3-1}}$$

$$\int_0^2 \frac{e^{2x}}{e^{2x}+1}dx = \int \frac{\cancel{e^{2x}}}{u}\cdot\frac{du}{2\cancel{e^{2x}}} = \int \frac{1}{2}\frac{du}{u} = \frac{1}{2}\ln|u|$$

$u = e^{2x}+1$

$du = (2e^{2x}+0)dx$

$\frac{du}{2e^{2x}} = dx$

$$\frac{1}{2}\ln|e^{2x}+1|\Big[_0^2$$

$$\frac{1}{2}\ln(e^{2\cdot 2}+1) - \frac{1}{2}\ln(e^{2\cdot 0}+1)$$

$e^0 = 1$

$1+1=2$

$$\frac{1}{2}\ln(e^4+1) - \frac{1}{2}\ln(e^0+1)$$

$$\frac{1}{2}\ln(e^4+1) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln\left|\frac{e^4+1}{2}\right|$$

$$\int_2^6 x^2\sqrt{x+3}\,dx = \int x^2\sqrt{u}\,du = \int_5^9 (u-3)^2 \cdot u^{\frac{1}{2}}\,du = \int_5^9 (u^2-6u+9)u^{\frac{1}{2}}\,du$$

$u = x+3$
$u-3 = x$
$du = dx$

$u = 2+3$
$u = 6+3$

$$\int_5^9 \left(u^{5/2} - 6u^{3/2} + 9u^{\frac{1}{2}}\right) du$$

$$\frac{2}{7}(x+3)^{7/2} - \frac{12}{5}(x+3)^{5/2} + 6(x+3)^{3/2}\Big|_2^6 = \frac{2}{7}u^{5/2+1=\frac{7}{2}} - 6\cdot\frac{2}{5}\cdot u^{3/2+1=\frac{5}{2}} + 9\cdot\frac{2}{3}\cdot u^{\frac{1}{2}+1=\frac{3}{2}}\Big|_5^9$$

$$\frac{2}{7}(6+3)^{7/2} - \frac{12}{5}(6+3)^{5/2} + 6(6+3)^{3/2}$$

$$\frac{2}{7}\left(u^{\frac{7}{2}}\right) - \frac{12}{5}u^{5/2} + 6u^{3/2}\Big|_5^9$$

$$\left(\frac{2}{7}(2+3)^{7/2} - \frac{12}{5}(2+3)^{5/2} + 6(2+3)^{3/2}\right)$$

$$\frac{2}{7}\left(9^{7/2}\right) - \frac{12}{5}(9)^{5/2} + 6(9)^{3/2} - \left[\frac{2}{7}(5)^{7/2} - \frac{12}{5}(5)^{5/2} + 6(5)^{3/2}\right]$$

$$9^{7/2} = \left(3^2\right)^{7/2} = 3^{\cancel{2\cdot\frac{7}{2}}} = 3^7$$

$$\frac{2}{7}\cdot 3^7 - \frac{12}{5}\cdot 3^5 + 6\cdot 3^3 - \frac{2}{7}5^{7/2} - \frac{12}{5}(5)^{5/2} + 6(5)^{3/2}$$

$$\frac{2}{7}(2187) - \frac{12}{5}(243) + 6\cdot 27 - \sqrt{5}\left(\frac{2}{7}\cdot 5^3 - \frac{12}{5}(5)^2 + 6\cdot 5\right)$$

$$203\,\frac{23}{35} - \sqrt{5}\left[5\,\frac{5}{7}\right]$$

$$\int_2^6 x^2\sqrt{x-1}\,dx = \int (u+1)^2\cdot u^{\frac{1}{2}}\,du = \int (u^2+2u+1)\,u^{\frac{1}{2}}\,du = \int \left(u^{5/2} + 2u^{3/2} + u^{\frac{1}{2}}\right) du$$

$u = x-1$
$u+1 = x$
$du = dx$

$$\frac{2}{7}u^{5/2+1=\frac{7}{2}} + 2\cdot\frac{2}{5}u^{\frac{3}{2}+1=\frac{5}{2}} + \frac{2}{3}u^{\frac{1}{2}+1=\frac{3}{2}}$$

$$\frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2}\Big|_2^6 = \sqrt{(x-1)}\left[\frac{2}{7}(x-1)^3 + \frac{4}{5}(x-1)^2 + \frac{2}{3}(x-1)^1\right]$$

$$\left[\frac{2}{7}(6-1)^{7/2} + \frac{4}{5}(6-1)^{5/2} + \frac{2}{3}(6-1)^{3/2}\right] - \left[\frac{2}{7}(2-1)^{7/2} + \frac{4}{5}(2-1)^{5/2} + \frac{2}{7}(2-1)^{3/2}\right]$$

$$\frac{1240\sqrt{5}}{21} - \frac{184}{105}$$

3.

$$\int_0^1 x^2 \cdot e^{x^3+1}\,dx = \int \cancel{x^2} \cdot e^u \cdot \frac{du}{3\cancel{x^2}} = \frac{1}{3}\int_1^2 e^u\,du = \frac{1}{3}e^u + C$$

$$\frac{1}{3}e^2 - \frac{1}{3}e^1$$

$u = x^3+1$ $\quad u = 1^3+1 = 2$

$du = 3x^2\,dx$ $\quad 0^3+1 = 1$

$\frac{du}{3x^2} = dx$

$$\frac{1}{3}e^{x^3+1}\Big[_0^1 = \frac{1}{3}e^{1^3+1} - \frac{1}{3}e^{0^3+1} = \frac{1}{3}e^2 - \frac{1}{3}e^1$$

$$\frac{e}{3}(e-1)$$

$$\int_0^1 x^3 \cdot e^{x^4+1}\,dx = \int_1^2 \cancel{x^3} e^u \cdot \frac{du}{4\cancel{x^3}} = \frac{1}{4}\int_1^2 e^u\,du = \frac{1}{4}e^u\Big[_1^2 du$$

$u = x^4+1$ $\quad u = 1^4+1 = 2$

$du = 4x^3\,dx$ $\quad u = 0^4+1 = 1$

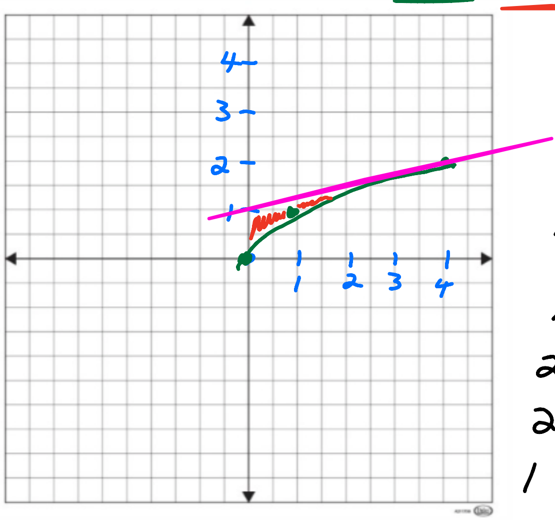
$\frac{du}{4x^3} = dx$

$$\frac{1}{4}e^2 - \frac{1}{4}e^1$$

$$\checkmark \frac{1}{4}e^{x^4+1}\Big[_0^1 = \frac{1}{4}e^{1^4+1} - \frac{1}{4}e^{0^4+1}$$

$$\frac{1}{4}e^2 - \frac{1}{4}e$$

4. Find the area enclosed by the curve $y = \sqrt{x}$, the *tangent* to the curve at $x = 4$, and the $y - axis$.



$y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ at $x = 4$ $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

Point $(4, 2)$ $\Rightarrow y = \frac{1}{4}x + 1$

$y = \frac{1}{4}x + b$

$2 = \frac{1}{4}(4) + b$

$2 = 1 + b$

$1 = b$

$\int_0^4 \left[\left(\frac{1}{4}x + 1\right) - \sqrt{x}\right] dx$

$\frac{1}{8}x^2 + x - \frac{2}{3}x^{3/2} \Big|_0^4$

$(\sqrt{4-x})^2 = (x+2)^2 = (x+2)(x+2)$

$4 - x = x^2 + 4x + 4 \Rightarrow 0 = x^2 + 5x + 0$

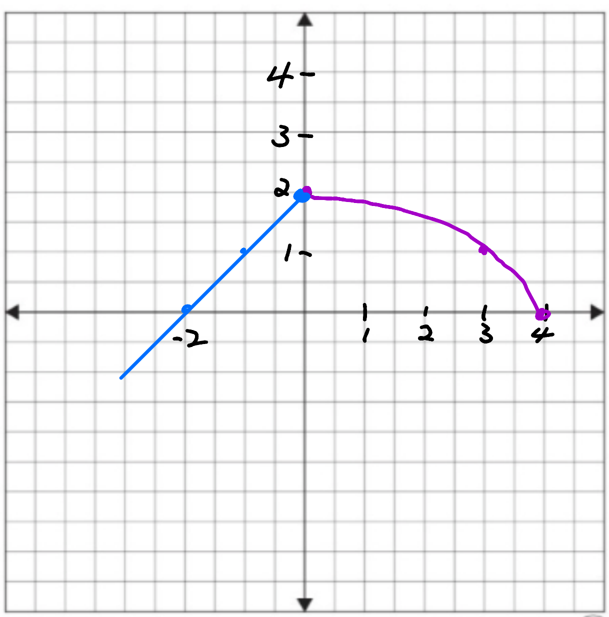
$x(x+5) \Rightarrow x = 0$ or 5

$y = \sqrt{4-x}$

$y^2 = 4 - x$ $\rightarrow$ $x = 4 - y^2$

$x = y - 2$

6. What is the area of the region bounded by the graphs of $f(x) = \sqrt{4-x}$ and $g(x) = x + 2$ and the x-axis?



$\int_{-2}^{0} (x+2)dx + \int_0^4 \sqrt{4-x}\, dx$

or

$\int_0^2 [(4 - y^2) - (y - 2)] dy$

$\int_0^2 (4 - y^2 - y + 2) dx$

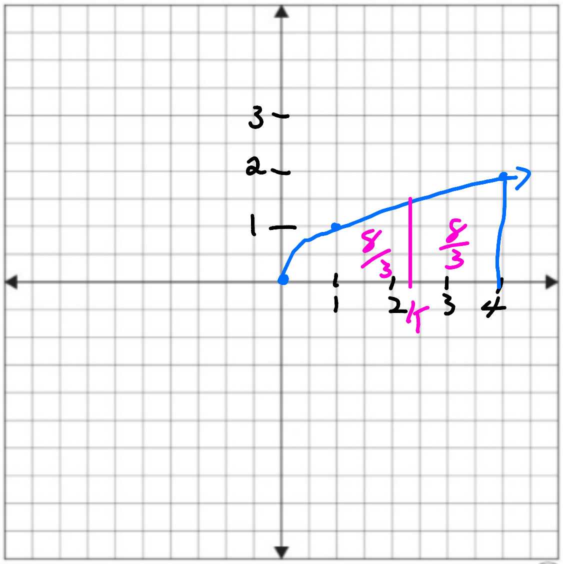
$\int_0^2 (6 - y^2 - y) dy$

$6y - \frac{1}{3}y^3 - \frac{1}{2}y^2 \Big|_0^2$

$6 \cdot 2 - \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 - \left[\cancel{6 \cdot 0 - \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2}\right]$

$12 - \frac{8}{3} - 2 =$

7. The vertical line $x = k$ divides the region enclosed by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$ into 2 regions of equal area. Find the value of k.



$$\int_0^4 x^{\frac{1}{2}} dx = \frac{2}{3} x^{3/2} \Big|_0^4$$

$$\frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3} - 0 = \frac{16}{3}$$

Half of $\frac{16}{3} \Rightarrow \frac{8}{3}$

$$\int_0^k x^{\frac{1}{2}} dx = \frac{8}{3}$$

$$\frac{2}{3} x^{3/2} \Big|_0^k = \frac{8}{3}$$

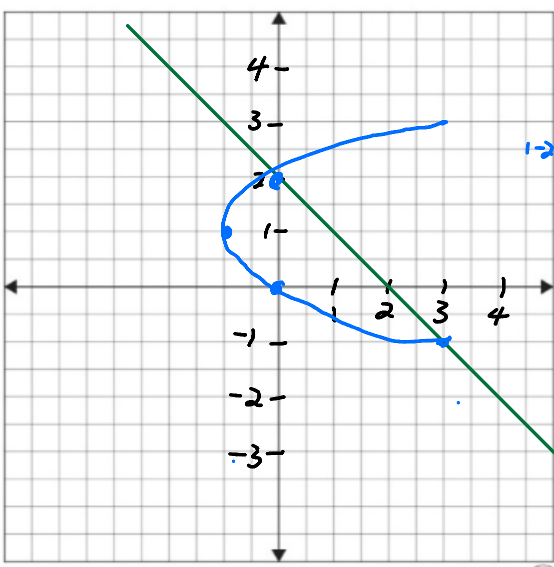
$$\frac{2}{3} \cdot k^{3/2} - \frac{2}{3} \cdot 0^{3/2} = \frac{8}{3}$$

$$\frac{3}{2} \cdot \frac{2}{3} k^{3/2} = \frac{8}{3} \cdot \frac{3}{2}$$

$$\left(k^{3/2}\right)^{\frac{2}{3}} = (4)^{2/3}$$

$$k = 4^{2/3} = 2^{4/3}$$

5. What is the area of the region bounded by the graphs of $x = y^2 - 2y$ and $y = -x + 2$?



x	y
0	0
0	2
$1-2=-1$	1
3	-1

$x = -y + 2$

$$\int_{-1}^{2} \left[(-y+2) - (y^2 - 2y)\right] dy$$

$$\int_{-1}^{2} \left[-y + 2 - y^2 + 2y\right] dx$$

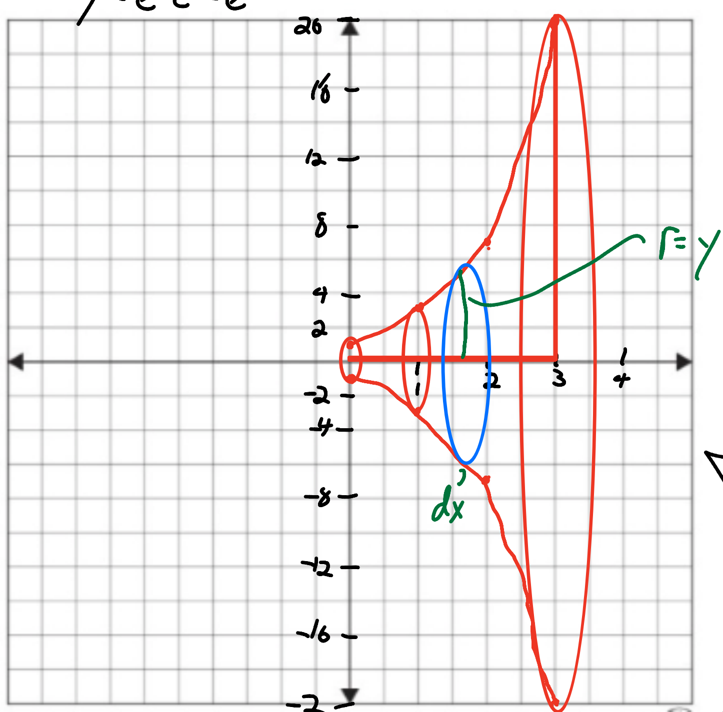
$$\int_{-1}^{2} \left[-y^2 + y + 2\right] dy$$

$$-\frac{1}{3} y^3 + \frac{1}{2} y^2 + 2y \Big|_{-1}^{2}$$

$$-\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2) - \left[-\frac{1}{2}(-1)^3 + \frac{1}{2}(-1)^2 + 2(-1)\right] = \text{keep going}$$

5. *for this question, do not use a calculator to evaluate the integral; do it by hand, but simplify as much as possible.*
What is the volume of the solid of revolution generated when the region in the first quadrant bounded by the graph of $y = e^x$, the x-axis, and the line $x = 3$ is revolved about the x-axis.

$y^2 = e^x \cdot e^x = e^{2x}$



$$\int_0^3 \pi (y)^2\,dx$$

$$\pi\int_0^3 e^{2x}\,dx = \frac{\pi}{2} \cdot e^{2x} \Big|_0^3$$

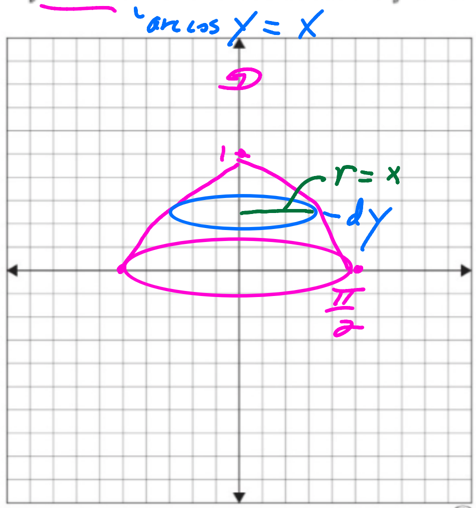
$u = 2x$

$\frac{du}{2} = dx$

$$\frac{\pi}{2}e^6 - \frac{\pi}{2}e^0$$

6. What is the volume of the solid of revolution generated when the region in the first quadrant bounded by $y = \cos x$ is revolved about the y-axis.

$\arccos y = x$

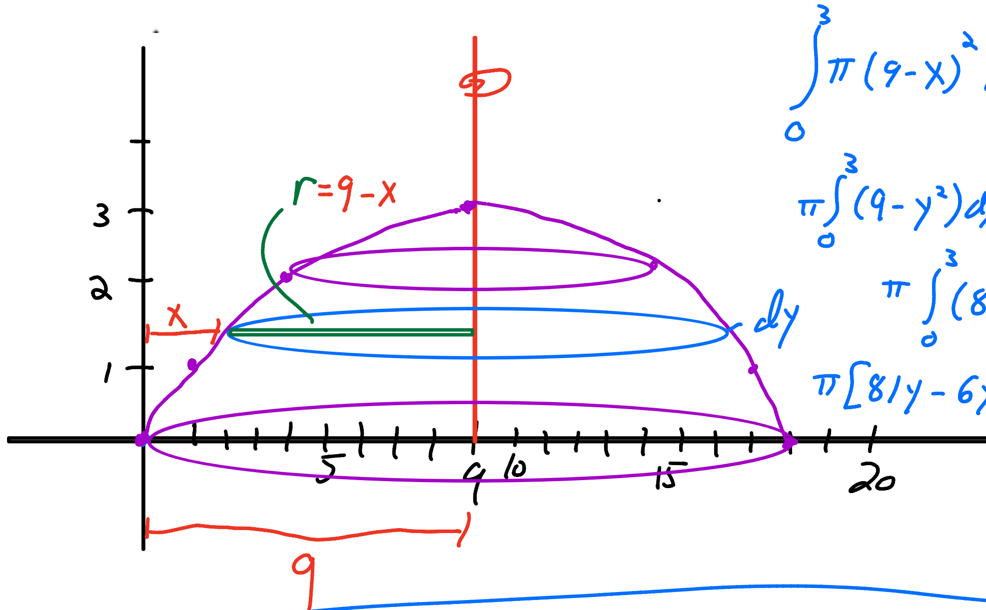


$$\int_0^1 \pi(x^2)\,dy$$

$$\int_0^1 \pi(\arccos x)^2\,dy = \pi(\pi - 2)$$

$y^2 = x$

3. Find the volume of the solid generated when the region bound by $y = \sqrt{x}$, $y = 0$, $x = 9$ is revolved about the line $x = 9$.



$$\int_0^3 \pi(9-x)^2\,dy$$

$$\pi\int_0^3 (9-y^2)\,dy$$

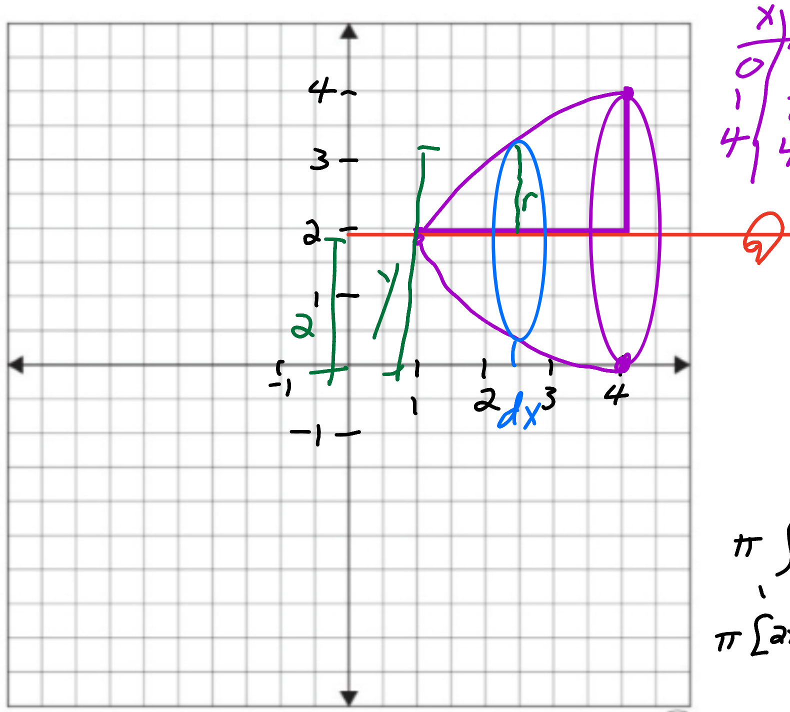
$$\pi\int_0^3 (81-18y^2+y^4)\,dy$$

$$\pi\left[81y-6y^3+\tfrac{1}{5}y^5\right]\Big|_0^3$$

4. Find the volume of the solid generated when the region bound in the first quadrant by $y = 2\sqrt{x}$, $x = 1$, $x = 4$, and $y = 2$ is revolved about the line $y = 2$.

$y = 2\sqrt{x}$ $\quad r = y - 2$

x	y
0	0
1	2
4	4



$$\int_1^4 \pi r^2\,dx$$

$$\int_1^4 \pi(y-2)^2\,dx$$

$$\pi\int_1^4 (2\sqrt{x}-2)^2\,dx$$

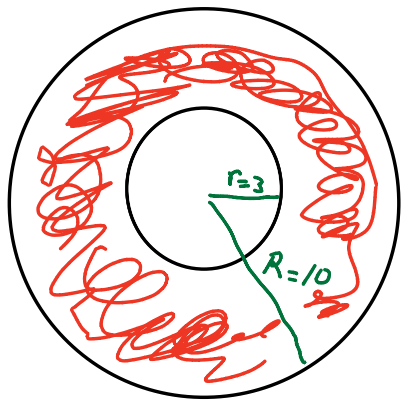
$$\pi\int_1^4 (4x-8\sqrt{x}+4)\,dx$$

$$\pi\left[2x^2-8\cdot\tfrac{2}{3}x^{3/2}+4x\right]\Big|_1^4$$

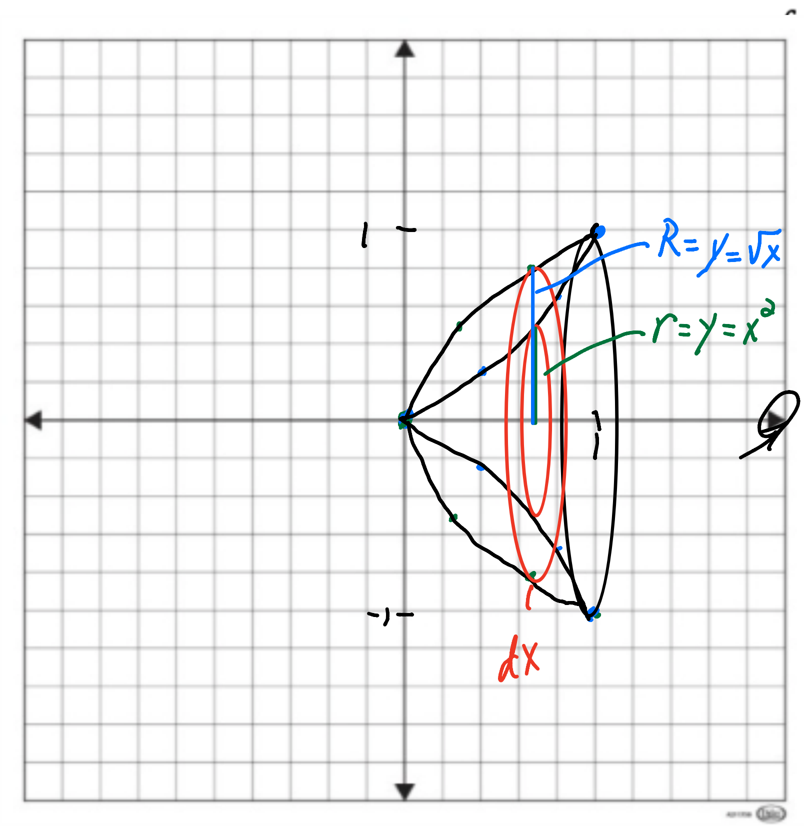
Washer Method

$10^2\pi - 3^2\pi = 100\pi - 9\pi = 91\pi$

$\pi(R^2 - r^2)$



Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$
About the x-axis



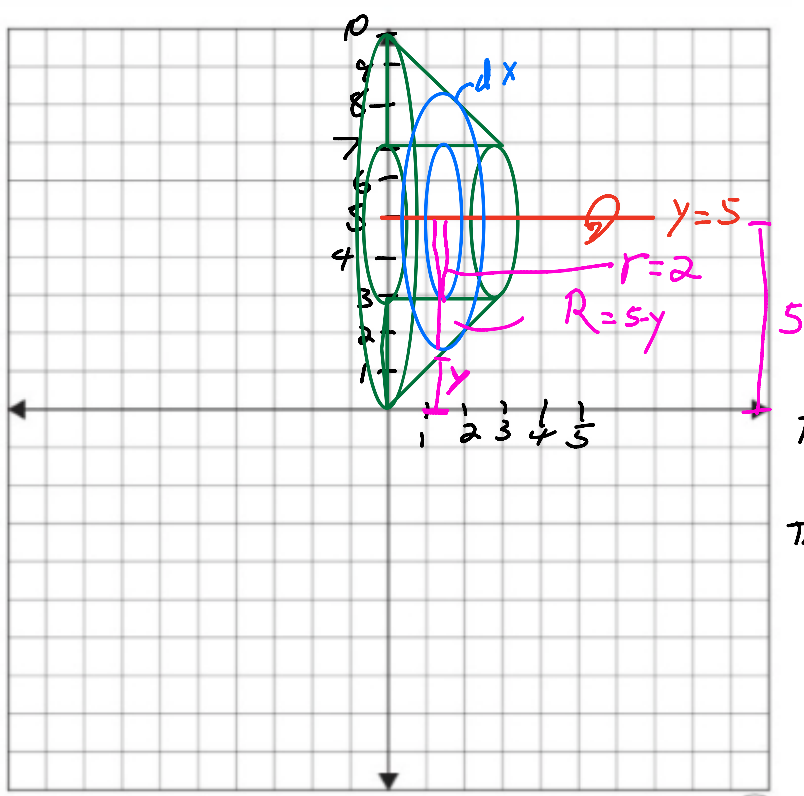
$$\int_0^1 \pi(R^2 - r^2)\,dx$$

$$\pi\int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right]dx$$

$$\pi\int_0^1 (x - x^4)\,dx$$

$$\pi\left(\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right)\Big|_0^1$$

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x$, $y = 3$, $x = 0$ about the line $y = 5$.



$$\int_0^3 \pi(R^2-r^2)\,dx$$

$\{x=y$

$$\pi\int_0^3 [(5-y)^2 - 2^2]\,dx$$

$$\pi\int_0^3 [25-10x+x^2-4]\,dx$$

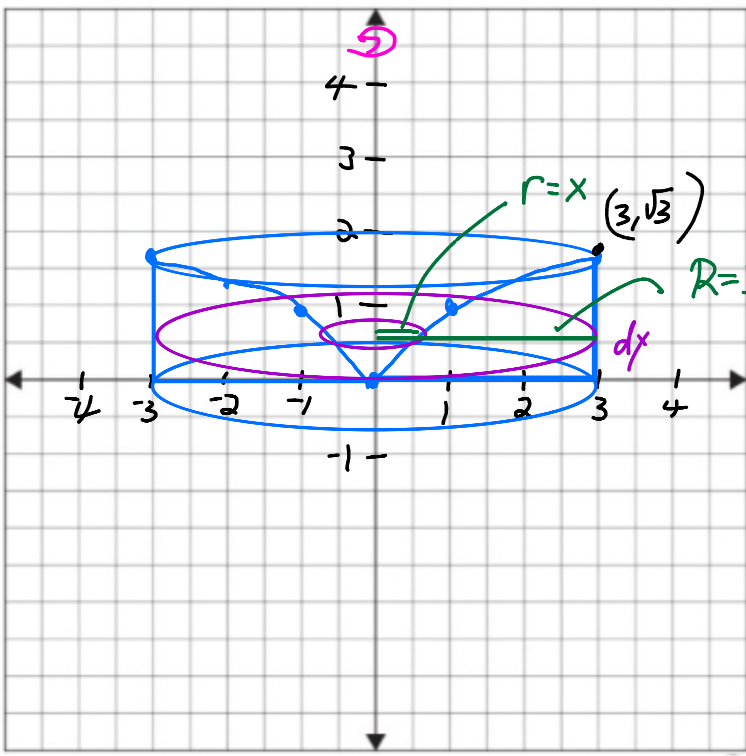
$$\pi\int_0^3 (21-10x+x^2)\,dx$$

$$\pi\left[21x-5x^2+\frac{1}{3}x^3\right]\Big|_0^3$$

Example 4

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 3$, rotated about the y-axis.

$y^2 = x$



$$\int_0^{\sqrt{3}} \pi(3^2 - x^2)\,dy$$

$$\pi\int_0^{\sqrt{3}} (9 - (y^2)^2)\,dy$$

$$\pi\int_0^{\sqrt{3}} (9 - y^4)\,dy$$

$$\pi\left[9x - \frac{1}{5}y^5\right]_0^{\sqrt{3}}$$